ANNALES

PROCEEDINGS OF THE ACADEMY OF SCIENCES OF BOLOGNA

Class of Physical Sciences

Bologna
University Press

ANNALES

PROCEEDINGS OF THE ACADEMY OF SCIENCES OF BOLOGNA

CLASS OF PHYSICAL SCIENCES

1

Annales. Proceedings of the Academy of Sciences of Bologna
Class of Physical Sciences

Fondazione Bologna University Press
Via Saragozza 10, 40123 Bologna
tel. (+39) 051 232 882 – fax (+39) 051 221 019
ISSN: 2975-2302
ISBN: 979-12-5477-324-6
ISBN online: 979-12-5477-325-3
DOI: 10.30682/annalesps2301

www.buponline.com
info@buponline.com

Cover: Pellegrino Tibaldi, *Odysseus and Ino-Leocothea*, 1550-1551,
detail (Bologna, Academy of Sciences)

Layout: Gianluca Bollina-DoppioClickArt (Bologna)

First edition: December 2023

Table of contents

Some specific historical elements on the evolution of "Mathematics Education" as a research discipline

Bruno D'Amore

Academy of Sciences of Bologna Institute; University Francisco José de Caldas, Bogotá; Corresponding Academician

Abstract

The discipline "Mathematics Education" is part, in the Italian university world, of the MAT 04 disciplinary scientific area; courses in this discipline are held in all Italian universities at various academic levels: Faculty of Science, degree courses for the training of primary school teachers, masters and research doctorates. In this conference the international lines of the historical evolution of this discipline are traced, with its antecedents, in an attempt to explain what the current conditions are.

Keywords

Mathematics Education, History of Mathematics Education, Mathematics Educations as Mathematics field, Mathematics Education as scientific theory.

Doi: 10.30682/annalesps2301c

Premise

Although very young in the panorama of disciplines belonging to the large family of Mathematics, so much that still many people do not know about it at all or confuse it with something else, Mathematics Education has achieved an international scientific status which allows it to be the topic of specific research and therefore congresses, evaluations, journals, analyzes by referees etc., in the same way as many other branches of Mathematics. It therefore seemed to us appropriate to outline an albeit brief history of this discipline as it is done for other disciplines related to Mathematics. This is the theme of the following short text in which we limit ourselves to underline the controversial birth and development of Mathematics Education in the mathematical world.

1. The interest of mathematicians in the problem of teaching/learning Mathematics

Interests of professional mathematicians in the problem of learning Mathematics have always existed; it is enough to recall the legend, according to which Euclid (4th-3rd century BC) drove Pharaoh Ptolemy I (367-283 BC) from his classroom because he wanted to learn geometry quickly and effortlessly…

We can then highlight some elements of the *Encyclopédie* (18th century) in which the authors are pushed to ask themselves: What does "simple to understand" mean? Is the "simple" an absolute or a relative fact? Is the "simple" the same for the scientist and for a child who learns? Or is there a difference? If so, which one? These questions find attempts of answers even by Jean-Baptiste Le Rond D'Alembert (1717-1783) and Denis Diderot (1713-1784), especially in the articles *Analysis*, *Synthesis*, *Method*, *Elements of science*. In our opinion, it is already a study of specific didactics that differs from the general interests of Pedagogy (that is: the specificity is due to the specificity of the involved object of knowledge). It seems strange and interesting to us how precisely this didactic debate makes D'Alembert pass from an entirely Cartesian position to a Lockean one and then how he tries to reconcile the two: "Simple ideas can be reduced to two species: one of abstract ideas […] The second species of simple ideas is contained in the primitive ideas that we acquire through our sensations".

2. The Lichnerowicz Commission, Modern Mathematics and the triumph of the language of sets

But let us get closer to our times. On April 12, 1961, during the Vostok 1 mission, Jurij Gagarin (1934-1968) for the first time in history made a revolution around the Earth; the whole world recognized the evident Russian superiority on a technological and therefore scientific level and, among the immediate consequences of this event, many countries decided to radically renew their school curricula, especially in regards to Mathematics.

In particular, in France, in 1967, the famous mathematician and physicist André Lichnerowicz (1915-1998) (French of Polish descent) was invited by the French government to preside over a commission whose task was to reformulate from scratch not only the national Mathematics school curricula, but also the themes and methods of its teaching. The commission became known as *the Lichnerowicz Commission*. It was composed of 18 mathematics teachers chosen from among those suggested by the various local communities. The commission, after a few months of work, recommended various issues, but above all a curriculum based entirely on the teaching of set theory, suggesting an early introduction to mathematical structures from the very beginning of primary school. This reform proposal was called *New Mathematics* and was accepted in many other countries around the world.

The mathematical inspiration behind these suggestions obviously refers to the so-called *Bourbakist movement*, active in France with great scientific success since 1935 (and certainly until 1983). Given the great prestige of the president of this commission, in France the recommendation was immediately accepted by mathematics teachers also because it was immediately endorsed by the ministry. Thus, the idea of radically changing the teaching of Mathematics starting from primary school spread throughout Europe and many other nations on various continents. This radical change focused everything on a "naïve" set theory, where a set was thought of as "collection" and similar, in a strictly concrete sense. Actually, this "race for sets" was only the emerging tip of a broader vision of basic Mathematics that went under various denominations in different countries: *New Mathematics*, *Modern Mathematics* and others.

The great fortune of the mentioned race for sets in school Mathematics is also and above all linked to the many prepared materials that accompanied it which were called "structured materials"; but also to the theories of the mathematician Zoltan Dienes (1916-2014) and the psychologist Jerome Bruner (1915-2016). We will say about the former much later. The second, in his *Theory of Education* (Bruner 1966), argues that the very structure of knowledge must be developed in students' mind. In particular, in Mathematics, one should not focus only on mechanical or algorithmic skills, nor limit oneself to giving simple information: One must structure the mind exactly as Mathematics itself is structured, in order to then be able to compose the individual pieces, within this structure already in place.

3. The proliferation of tools for teaching Mathematics

As mentioned above, the infatuation with Modern Mathematics prompted many authors to create tools of various kinds considered as effective, indispensable, infallible materials for teaching Mathematics.

We recall only the most famous, spread throughout the world, and their specific theme:

– logical blocks by Zoltan Dienes: teaching of the logic of sets;

– colored numbers by Caleb Gattegno (1911-1988), but made by Georges Cuisinaire (1891-1975): basic elements of arithmetic;

– Georges Papy's minicomputer: automatic switch from base ten to base two;

– 4-string musical instrument by Zoltan Dienes to "experience Algebra with bodily actions";

and many others, whose production (with illusory effects and often containing mathematical errors) still continues.

Let us see some general critical notes which, however, were highlighted only between the years '75 and '86 by the studies of Guy Brousseau, as we will highlight in the following.

1. Dienes' logical blocks were made up of elementary geometric figures of different shapes, different sizes and different colors. Let us suppose the child learns, for example, that the intersection between the set of yellow blocks and the set of round blocks is made up of the set of round yellow blocks; this does not mean that he/she learned Logic or even just the intersection between sets. Indeed, it can easily be verified that, by changing the example, abandoning the blocks of that specific box, and moving on to another example (to numbers, concrete objects, or something else), the child does not recognize the situation which for the adult is obviously analogous. He/she has learned the intersection *in that given context* but does not know how to generalize it and therefore use it in other contexts; as experienced and critical teachers commented, the child learns "in a sealed chamber". Better said: *Cognitive transfer* does not occur automatically because true conceptual learning requires the abstract shift from learning in certain circumstances to general conceptualization, thanks to the teacher's didactic competence. Conversely, the inventors of these games believe that this generalizing shift is spontaneous, as it could be for an adult. The same teacher is displaced, he does not know exactly what is expected of the children and their learning: He/she follows the advice of the great, celebrated, and famous mathematics educator and does not know how to recognize learning that has taken place. Or even worse: He ends up considering the words and behaviors that he knows are the expected ones, as if they were genuine learning, but without even knowing why.
2. Dienes had invented a motto, *la mathématique vivante*, in French. It was a question of making the pupils *act* concretely (Dienes 1972). In his opinion, by doing, acting, in a physical, bodily way, they would have assimilated and learned even without wanting to, without realizing it. (Exactly the opposite of what Guy Brousseau, the one who denounced all this construction and who created modern research in Mathematics Education, would assert a few years later: There is no learning without awareness, without an explicit will to engage in learning.) Dienes had created a 4-string musical instrument that he used to playfully create algebraic structures implicit in the activity that the child learned and made his own with appropriate games, actions, and movements. This instrument, according to him, guaranteed the implicit formation of skills and knowledge that could then be used in school practice.

Caleb Gattegno invented, and then George Cuisinaire created, nice colored prisms starting from a unitary white cube which represented the unit (1) up to a unitary base and light brown prism obtained by superimposing 10 of these cubes of another color which represented the 10. The small child, first years of primary school, instead of working on the numbers themselves (denounced as "too abstract") worked directly with these colorful attractive objects. [Why then it should be believed that it is easier to understand the operation "green plus yellow equals brown" rather than "3+5=8", still remains a mystery]. Yet this nonsense invaded the whole world and for decades children played with these colorful objects. Of course, no cube or small prism could represent zero, which was thus excluded from the arithmetic of natural numbers. For example, it was not permitted to subtract 5–5.

Georges Papy became internationally famous for his chromatic rules relating to how to draw those oval-shaped graphs with which sets are represented. Mostly, teachers were happy to follow such rules, far from any semiotic or mathematical sense, dictated by such a famous character, easy to understand and well-liked by children, always happy to learn understandable rules and use colors.

Papy was also famous for a so-called mini-calculator that bore his name, which helped to make calculations in the binary basis, nothing more than a small cardboard square divided by the medians into 4 smaller squares of 4 different colors. This ludic tool easily transformed writings in base ten into writings in base two, without even understanding the meaning, performing pure automatic transformations of writing.

Dienes and Papy traveled the whole world, all continents, triumphantly presenting their playful creations. And spreading *Modern Mathematics*.

4. The condemnations of Modern Mathematics by important mathematicians

But since 1970 strong signs of rejection of these didactic hypotheses began to circulate.

In 1970 the famous article by the mathematician René Thom, *Modern Mathematics: an educational and philosophical error*? (Thom 1970) was published. It should be remembered that in 1958 Thom had won the prestigious Fields Medal; therefore, his entry into the field had a not inconsiderable weight. This article contained, in very few pages, a very critical analysis that suddenly reawakened the interest of mathematicians on the problems of mathematical education. Subsequently, in 1972, Thom himself vehemently reiterated his thinking at the II International Congress on Mathematical Education held in Exeter, England (Thom 1973).

A further proud blow came from another famous character, the American historian of mathematics Morris Kline (Kline 1973). The work, entitled: *Why can't Johnny add*?, had an explicit subtitle: *The failure of the New Mathematics*.

After the attack by mathematicians came the attack by psychologists who did not consider the Piagetian number theory, which many held as the basis of Modern Mathematics, convincing at all. But let us skip over this point.

Since the mid-1980s, in contrast with the New Mathematics, various educational projects have been born all over the world and, above all, a new vision of research in the teaching of arithmetic and number was beginning to take hold. They have given new vitality to the ordinal idea, to the number within recursion, but also to the number thought of as a term of the language for naming, to the number in its temporal meanings, in the use of money, etc.

Many of these requests have been collected by the various nations, at the time of reformulating the national arithmetic curricula, especially for elementary schools. For example, in the Italian school, the national curricula of 1985 mention, instead of sets, under the heading *Arithmetic*: "The development of the concept of number must be stimulated by making the most of the pupils' previous experiences in counting and recognizing numerical symbols, made in the context of play and family and social life. It should be kept in mind that the idea of natural number is complex and therefore requires an approach that makes use of different points of view (ordinality, cardinality, measure, etc.); its acquisition takes place at increasingly higher levels of internalization and abstraction throughout the entire elementary school course, and beyond" (D.P.R. 104, p. 24, translation by the author).

It is obvious that these words were inspired by the previous story.

The same can certainly be said for the historic turning point of French curricula, even those of 1985 and currently no longer in force, which previously were all devoted to set theory and which are now totally devoid of *it*. Moreover, no reference to ensembles appears in the British national curricula (of 1988) which were above all indications of minimum learning goals.

5. A specific case: Italy and Modern Mathematics

In 1974 the Italian Ministry of Education created the Regional Institutes of Educational Research (*IRRE*) as a concrete expression of the desire/need strongly felt in those years to renew all aspects of teaching in Italy, not only from an administrative bureaucratic point of view, but also in reference to the specific disciplines. The concrete expression of the *IRRE* were the *Regional Institutes for Research, Experimentation and Educational Updates* (*IRRSAE*), one for each region. Zoltan Dienes was appointed director of the *IRRE* of the Emilia-Romagna region.

That of the *IRRSAE* was a moment of great euphoria, which went from the mid-1980s until 2005. Training courses began to be held in all regions, creating experimental groups with the intention of radically renewing the practices in teaching Mathematics from all points of view. This led to the need for the so-called *New Elementary School Programs*, the new primary school curricula, issued in 1985, as we have already mentioned.

Above all, thanks to concrete experiences and experimentations, various totally new specific projects came to light, intended for the training of primary school teachers, for example the *Ma.SE Project* (*Elementary School Mathematics*). Several volumes were published, especially between 1986 and 1989.

6. The Cognola event and the first signs of the Italian revolt against Modern Mathematics

In 1908, the International Commission on Mathematical Education (ICMI) was created, during the International Congress of Mathematicians. In 1955 the Italian Commission for the Teaching of Mathematics appears in Italy, in 1963 the acronym CIIM is officially present within the Italian Mathematical Union and in 1975 it becomes a permanent commission of the UMI.

These dates show a real interest of mathematicians and scientists towards the school and the school teaching.

There were not many university researchers who decided to deal with this kind of problems interpreting them as real scientific research, but not even very few.

To give a sign of vitality and to strengthen interest in the field, the Italian Mathematical Union [at the time the Florentine analyst Carlo Pucci (1925-2003) was president] decided to hold, with funds from the *CNR*, an international conference entitled *Cognitive processes of mathematics at the primary school level*. The conference took place in Cognola (Trento), from 7 to 11 October 1980. As speakers were invited the most famous foreign scholars of educational problems relating to the teaching of Mathematics at the time [among them: Zoltan Dienes, George Papy, Fredérique Papy (1921-2005), Zofia Krygowska (1904-1988) and Efraim Fischbein (1920-1998)], but also young Italian mathematicians active in research in Mathematics at universities, but interested, or at least intrigued by, the topic of his teaching in schools, were invited not only as listeners, but also to hold some reports.

Dienes and Papy, finally in front of critical young scientists, revealed their scientific and methodological weaknesses as researchers; while Efraim Fischbein impressed all the Italians present, so much so that then, for years, he collaborated with many Italian educational research centers to carry out much appreciated research. He had founded *PME* (The *International Group for the Psychology of Mathematics Education*) in 1976 during the *ICME* (*International Congress on Mathematical Education*) 3 in Karlsruhe. Fischbein had conceived, built and then launched as possible research (in the specifically didactic field) the so-called *theory of figural concepts*, fascinating and convincing. It was conceived in the early 60s and published in Romanian in 1963 (Fischbein 1963), ignored by the academic world; it was translated into English only 30 years later (Fischbein 1993), obtaining the international success it deserved (and deserves).

As stated before, at an international level, since 1908 there was an International Commission on Mathematical Education founded in Rome, and Felix Klein (1849-1925) was elected as its first president; while the *ICME* was born in 1969 on the initiative of the president of the *ICMI* Hans Freudenthal (1905-1990) who had a notable interest in the teaching of Mathematics at school. In other words, the attention of mathematicians towards school Mathematics has always existed, but not yet in the current direction, of considering it in turn as a possible scientific research domain within mathematics itself.

In 1992, the entire national conference *Incontri con la Matematica 6*, held in Castel San Pietro Terme (Bologna), from 13 to 15 November, had as its only speakers Efraim Fischbein and Gérard Vergnaud (1933-2021), another great scholar in Mathematics Education. This can be considered as a sign of the fact that in the field of research in Mathematics Education everything had changed, compared to *Modern Mathematics*: Finally, it was possible to speak of real scientific research.

7. Guy Brousseau and the birth of Mathematics Education

From the earliest years of *New Mathematics*, a French primary school teacher, Mathematics student and then university professor of mathematics in Bordeaux, Guy Brousseau, fought vigorously against that distorted form which had imposed itself on the world. Indeed, Brousseau proposed the possibility of serious scientific mathematical research, in the educational-scholastic field.

Over the years, his research and publications reached the various research groups that were slowly forming in the world, and therefore the teachers; we have seen above all what happened in Italy, but the situation was similar in many other countries.

We believe that the most significant and definitive text was the one published in 1986 (Brousseau 1986) finally in a real specific journal that bears the name of the discipline he conceived and called: *Recherches en Didactique des Mathématiques* (*Investigations in Mathematics Education*) born in 1980.

Brousseau was undoubtedly the creator of theses (all linked to mainly empirical research results) which formed the basis on which that scientific discipline is founded which is currently called Mathematics Education, no longer to be confused with questions of common sense, personal ideas, ideas related to teaching, devising new ways of teaching, creating toys suitable for teaching, etc. The difference and the relationship between teaching and learning are now a common domain entirely shared by those involved in research, and not just proposing ideas based on their common sense or on their intuitions or opinions.

We do not think it is appropriate to attempt to give a nod to Brousseau's profound, revolutionary research results in a few lines. But we would like to point out that the *first* Felix Klein Medal of the International Commission on Mathematical Education was awarded to him in 2003.

We limit ourselves to making a list of the main themes on which he based his scientific research discipline: *theory of situations*, *obstacles*, the *didactic triangle model*, *didactic transposition*, *didactic engineering*, and others. Hundreds, indeed thousands, of researchers in Mathematics Education from all over the world then worked on these first topics obtained as research results by Brousseau, confirming and deepening these first scientific elements intuited, proposed, defined by Brousseau. These are those topics that constitute the basic structure of research today, for example by guaranteeing journal referees anchors and references for issuing judgments that are not just an expression of personal ways of thinking or judging.

8. The definitive decline of the ideas of Dienes, Papy and Modern Mathematics

Since 1970 George Papy had created, and directed as president until 1991, a study and research group, the *GIRP* (*Groupe International de Recherche en Pédagogie de la Mathématique*) with official headquarters in Walferdange (Luxembourg). During the conference in 1991 in Locarno, another president was elected, and in 1994 the *GIRP* dissolved.

Dienes' fame still persisted in the early 1990s in various areas of Italy and especially in Romagna, above all thanks to a group of educational directors who had been trained by him in the region. They organized in Forlì, on May 8th, 1993, a meeting-debate between Dienes himself and an exponent of the new ideas of Mathematics Education created in France, but now spread all over the world. In that occasion, it was Dienes himself who recognized the decline of his ideas.

So, both, the fanatical rush towards naïve set theory from primary school and *Modern Mathematics*, disappeared, in a certain sense rapidly. But those sneaky illusory tricks whose efficacy still too many teachers believe today, and which are still conceived, created and marketed by pseudo teaching experts, did not disappear. The *CIIM* and other scientific organisations of great importance have proposed criticisms of these silly tricks (often also full of mathematical errors); but those teachers who have never taken serious training courses in Mathematics Education are easy prey to these vapid illusions.

9. Other entities and events in Italy

Funded initially from the *CNR* and then also from the MIUR, the early 1980s also saw the birth of the NRD groups – Nucleo di Ricerca (Units of Investigation) in Mathematics Education which had the Mathematics Departments of Italian universities as their headquarters. The meaning was to create links for analysis and research between mathematicians teaching at universities and willing groups of schoolteachers (of all school levels), interested in research and not just in experimentation. The enthusiasm was remarkable, and it was an idea that we do not hesitate to define as successful. It was an excellent opportunity to make Italian and foreign school teachers aware of the research results of the then well-established discipline Mathematics Education with regard to the banalities listed above. We also highlight the *Internuclei* meetings, occasion for the exchange of information on research results between the different NRDs. Dozens of them were organized, all over Italy, and all of very high interest, with great participation of not only of university teachers, but also of schoolteachers.

Many activities carried out in this area were finally published as actual research results. Also, within the NRD, research journals were born in this specific field, which spread among teachers and are still active today. [Let us only recall: *L'insegnamento della matematica e delle scienze integrate*, since 1978 (it already existed with a very similar title, since 1970); *La matematica e la sua didattica* (since 1986)].

Various national conferences were born with the aim of making known the research results of the Nuclei and of disseminating and discussing the results of successful teaching experiments. Such conferences were therefore created not so much and not only for university researchers, but for all mathematics teachers; here we only mention the national conference *Incontri con la Matematica* (*Meetings with Mathematics*) which began in 1986 (edition number 0 in Bologna, from number 1 to the present, 2023, edition XXXVII, in Castel San Pietro Terme). It was an opportunity to introduce the most prestigious and most interesting international researchers to the audience of Italian teachers.

The international proliferation of similar activities, magazines and conferences, was very remarkable, always starting in the 80s, so much so that today they are countless, all over the world and in every season.

For many years, the two-year post-graduate courses (active from 1999 to 2010) called *SSIS* (*Schools of Specialization for Secondary Education*) were formidable and successful vehicles in Italian universities, intended for the training of future mathematics teachers in secondary schools, a model that convinced the whole world; while for future kindergarten and primary school teachers, specific five-year qualifying degree courses were set up at the faculties of education sciences, including actual courses in Mathematics. Both achieved good results, but only when the university teachers working there were really experts in Mathematics Education. The fact is that some colleagues ignore the story traced up to now and confuse research in Mathematics Education with common sense or with their own (sometimes naive) ideas, certainties, intuitions. In addition: some colleagues even ignore that there is real scientific research in this domain. Even more: there are many who confuse Mathematics Education with mathematics dissemination. Dissemination does not hurt, if it is well done and serious, but we should just not confuse it with what it is not. *SSIS* was abolished in 2008.

10. Brief notes on Brousseau's studies as a foundation for understanding Mathematics Education as a scientific theory

If a single but significant name can be given to the research started by Brousseau and then became fundamental throughout the world, so much so as to characterize those many initial research in a single theoretical way, we believe the one proposed by Brousseau himself fits perfectly: the "theory of situations". It is a very complex multiple theory, which then slowly evolved in ways and directions that today can also be called new theories, evolutions of that discipline to which everyone finally gives the same name: Mathematics Education, despite some variations on the meaning of which we will not go into detail here (Didactique des Mathématiques, Didattica della Matematica, Educación matemática, Matemática educativa, Educaçao matemática, Mathematikdidaktik…).

We are more than convinced that today Mathematics Education must be accepted as a scientific theory given that it has all the characteristics, especially in research. However, it does not seem appropriate here to deal with the following theme: Which characteristics must a theory have in order to be called a "scientific theory"? We shall limit ourselves only to referring to well-known authors, quoting famous theses and texts:

– paradigm and revolutions [Thomas Kuhn (1922-1996), (1957)]
– science as a specific system of knowledge [Mario Bunge (1919-2020), (1960)]
– research program [Imre Lakatos (1922-1974), (1978)]
– science as an expression of a coherent specific society [Thomas Albert Romberg, (1983)]
– science as a coherent set of principles, research questions and methodology [Luis Radford (Radford, 2021)].

We will not go into details, assuming that the authors cited here and their positions are very widespread.

Furthermore, we recall once again that, in the Italian ministerial list of university mathematical disciplines, "Mathematics Education" appears in the scientific disciplinary area "Mat 04 Complementary Mathematics" which also includes History of Mathematics and Elementary Mathematics from a higher point of view.

We note that theories constructed after the theory of situations often have different goals; they have been accepted with interest and curiosity in the international research panorama, but they have almost never been proposed to replace previous theories because these new theories almost always have different objectives. This theme, that of the relationships between theories or research that highlights analogies or contrasts between empirical research results, is one of the most fascinating topics of current research, in our opinion (D'Amore and Fandiño Pinilla 2017).

11. Mathematics Education as Applied Mathematics

Mathematics Education is therefore a discipline pertaining to the vast domain of Mathematics, in our opinion one of the many components of the so-called "Applied Mathematics", therefore a branch of Mathematics that deals with the study of mathematical techniques used in applying mathematical knowledge to other scientific and technical fields. This constitutes for many a further confirmation as regards the surprise of the power and success of applied mathematics, manifested for example by the Hungarian naturalized American physicist and mathematician Eugene Paul Wigner (1902-1995), in his famous and valuable popular book, published the first time in 1960, which has the significant title: *The unreasonable effectiveness of mathematics in the natural sciences* (Wigner 1960). We also suggest, in confirmation of our position, the illuminating reading of a very brief specific history of Applied mathematics (Stolz 2002).

A good example of the official recognition of this way of seeing by researchers in applied mathematics was, in our opinion, the *Joint Meeting of UMI–SIMAI/SMAI–SMF: Mathematics and its Applications*, held at the Mathematics Department of the University of Turin in July 2006. The idea (which we immediately found brilliant) is certainly due to Ferdinando Arzarello: a specific sector of that meeting was dedicated to the applications of Mathematics to the *Mathematics Education*, perhaps the most followed sector on that occasion, probably due to the curiosity that this new entry aroused among the participants. The texts of the conferences and seminars held by the various guests were published in full in number 1, volume 21, of the trilingual journal *La matematica e la sua didattica* (vol. 21, n. 1, April 2007) (D'Amore and Fandiño Pinilla 2007).

Over the decades, obviously many theoretical studies have followed one another, above all based on various results of empirical research and/or on specific fields of study and analysis. Of course, there are already many theoretical studies and also empirical research aimed at highlighting the differences between these theories. But new theories are born with precise objectives, not only to absorb or include previous theories, but also to study factors that the previous theories missed or to study facts in which previous theories were not interested (D'Amore 2007).

Conclusion

The details of entering the scientific world of Mathematics Education could be very detailed, given the numerous contributions to research that come from all over the world, from many universities and other research centers. This could be the theme of a new, much more extensive work, in which, for example, the characteristics of the different evolutionary lines that have followed one another could be highlighted. But the purpose of this text is summary, nothing more than an attempt to highlight the characteristics of the birth of a theory that first had to deal with unscientific premises based on activities and ideologies, rather than on scientific analytical reflections.

References

Brousseau, Guy. 1986. "Fondements et méthodes de la didactique des mathématiques". *Recherches en Didactiques des Mathématiques* 7 (2): 33-115.

Bruner, Jerome S. 1966. *Towards a Theory of Instruction*. Cambridge: Cambridge University Press.

Bunge, Mario. 1960. *Science, its method and its philosophy*. Buenos Aires: Eudeba.

D'Amore, Bruno. 2007. "Mathematical objects and sense: how semiotic transformations change the sense of mathematical objects". *Acta Didactica Universitatis Comenianae* 7: 23-45.

D'Amore, Bruno, and Pinilla Fandiño, Martha Isabel. 2007. "How the sense of mathematical objects changes when their semiotic representations undergo treatment and conversion". *La matematica e la sua didattica* 21 (1): 87-92. (Proceedings of: Joint Meeting of UMI–SIMAI/SMAI–SMF: *Mathematics and its Applications*. Panel on Mathematics Education. Department of Mathematics, University of Turin. July 6, 2006).

D'Amore, Bruno, and Pinilla Fandiño, Martha Isabel. 2017. "Reflexiones teóricas sobre las bases del enfoque ontosemiótico de la Didáctica de la Matemática - Theoretical reflections on the basis of the onto-semiotic approach to Mathematics Education". In *Actas del II Congreso Internacional Virtual sobre el Enfoque Ontosemiótico*. Granada, March 23-26, 2017, edited by J.M. Contreras, P. Arteaga, G.R. Cañadas, M.M. Gea, B. Giacomone and M.M. López-Martín. São Paulo: Proem Editora. http://enfoqueontosemiotic.ugr.es/civeos.html.

Decreto del Presidente della Repubblica 12 02 1985, n. 104 (D.P.R. 104). 1985. "Approvazione dei nuovi programmi didattici per la scuola primaria". Gazzetta ufficiale, 76. Disponibile da https://www.edscuola.it/archivio/norme/decreti/dpr104_85.pdf.

Dienes, Zoltan. 1972. *La mathématique vivante*. Paris: OCDL.

Fischbein, Efraim. 1963. *Conceptele Figurale*. Bucuresti: Editura Academiei RPR.

Fischbein, Efraim. 1993. "The Theory of Figural Concepts". *Educational Studies in Mathematics* 24 (2): 139-162.

Kuhn, Thomas S. 1957. *The Copernican revolution. Planetary astronomy in the development of western though*. Cambridge: Cambridge Harvard University Press.

Kline, Morris. 1973. *Why Johnny Can't Add: The Failure of the New Math*. New York: St. Martin's Press. https://doi.org/10.1177/019263657405837820.

Lakatos, Imre. 1978. *Philosophical Papers*. Cambridge: Cambridge University Press.

Radford, Luis. 2021. *The theory of objectification. A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Leiden, The Nederland: Brill/Sense. http://dx.doi.org/10.1080/10986065.2021.1984070.

Romberg, Thomas. 1983. "Toward "normal science" in some mathematics education research". *Zentralblatt für Didaktik der Mathematik* 15 (8): 89-92.

Stolz, Michael. 2002. "The History of Applied Mathematics and the History of Society". *Synthese* 133 (1): 43-57. http://dx.doi.org/10.1023/A:1020823608217.

Thom, René. 1970. "Les mathématiques modernes: une erreur pédagogique et philosophique?". *L'Age de la Science* III (3): 225-242.

Thom, René. 1973. Modern mathematics: does it exist? In *Developments in Mathematical Education*, edited by Albert G. Howson, 194-210. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9781139013536.011.

Wigner, Eugene. 1960. "The unseasonable effectiveness of Mathematics in the Natural sciences". *Communications on Pure and Applied Mathematics* 13, no. 1 (February): 1-14. https://doi.org/10.1002/cpa.3160130102.